Math 2270, Final

December 13, 2021

PRINT YOUR NAME: ______________________________

Question	Points	Score
1	20	
2	8	
3	10	
4	8	
5	8	
6	10	
7	10	
8	10	
9	8	
10	8	
Total:	100	

- No advanced calculators or cell phones or other electronic devices allowed at any time.
- Show all your reasoning and work for full credit, except where otherwise indicated.
- You have 120 minutes and the exam is 100 points.
- You do not need to simplify numerical expressions. For example leave fractions like 100/7 or expressions like $\ln(3)/2$ as is.
- Do the best you can!

1. (20 points)

1. Is the quadratic form $Q(x_1, x_2)$ corresponding to the matrix A positive definite, negative definite, or indefinite?

$$A = \begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix}$$

i) Positive definite ii) Negative definite iii) Indefinite

2. The determinant of a matrix B is $\det B = -13$. Is it invertible?

i) Yes ii) No iii) Not enough info

3. Let A be a 4×4 square matrix. Does it have a complex eigenvalue $\lambda = a+bi$? (b can be 0.)

i) Yes ii) No iii) Not enough info

4. What is the angle θ between the two vectors $\vec{u}$ and $\vec{v}$?

$$\vec{u} = \begin{bmatrix} \sqrt{6}+\sqrt{2} \\ -2\sqrt{2} \\ \sqrt{6}-\sqrt{2} \end{bmatrix}, \qquad \vec{v} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}.$$

$\theta =$ ______________

5. Is the linear transformation corresponding to the matrix A injective?

$$A = \begin{bmatrix} -133 & 211 & 186 \\ 190 & 210 & 1117 \end{bmatrix}$$

i) Yes ii) No iii) Not enough info

6. The columns of a matrix A are linearly independent. Is A surjective?

i) Yes ii) No iii) Not enough info

7. Find the projection $\mathrm{pr}_{\vec{v}}\vec{u}$ of $\vec{u}$ onto $\vec{v}$:

$$\vec{u} = \begin{bmatrix} -14 \\ 3 \\ 5 \end{bmatrix}, \quad \vec{v} = \begin{bmatrix} -1 \\ -2 \\ 2 \end{bmatrix}$$

$$\mathrm{pr}_{\vec{v}}\vec{u} = \begin{bmatrix} \_\_\_\_ \\ \_\_\_\_ \\ \_\_\_\_ \end{bmatrix}$$

8. Are these vectors orthonormal, orthogonal but not orthonormal, or neither?

$$\left\{ \begin{bmatrix} 2/\sqrt{5} \\ 0 \\ 1/\sqrt{5} \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} -1/\sqrt{5} \\ 0 \\ 2/\sqrt{5} \end{bmatrix} \right\}$$

i) Orthonormal ii) Just Orthogonal iii) Neither

9. Let A be a symmetric 6×6 matrix. Does there exist a basis for $\mathbb{R}^6$ consisting *only* of eigenvectors for A?

i) Yes ii) No iii) Not enough info

10. The characteristic polynomial $p_A(\lambda)$ of a 4×4 matrix A is

$$p_A(\lambda) = (\lambda - 1)(\lambda - 2)(\lambda - 3)(\lambda - 4).$$

Is A diagonalizable?

i) Yes ii) No iii) Not enough info

2. (8 points) Solve the system of linear equations.

$$\begin{aligned} 3x_1 - x_2 + x_3 &= -13 \\ x_1 \qquad\quad + x_3 &= -3 \\ -3x_1 - 3x_2 - 8x_3 &= -5 \end{aligned}$$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \underline{\qquad\qquad} \\ \underline{\qquad\qquad} \\ \underline{\qquad\qquad} \end{bmatrix}$$

3. (10 points) Find a basis for the null space $\operatorname{Nul} A$ and column space $\operatorname{Col} A$ of the matrix A. What are the dimensions of each space?

$$\begin{bmatrix} 2 & 2 & 0 & -2 \\ 10 & 9 & 2 & -7 \\ 3 & 3 & 0 & -3 \end{bmatrix}$$

$\operatorname{Nul} A$ basis: ______________________ $\dim \operatorname{Nul} A$: __________

Column Space $\operatorname{Col} A$:

$\operatorname{Col} A$ basis: ______________________ $\dim \operatorname{Col} A$: __________

4. (8 points) Find the determinant of the matrix A. Is it invertible?

$$A = \begin{bmatrix} 0 & 0 & 2 & 1 \\ 4 & 1 & 1 & 0 \\ 3 & 1 & 4 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

det $A =$ __________________

5. (8 points) Find the inverse A^{-1} of the matrix A:

$$A = \begin{bmatrix} 4 & 3 & 0 \\ 1 & 1 & 0 \\ 1 & 4 & 1 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} \rule{2cm}{0.4pt} & \rule{2cm}{0.4pt} & \rule{2cm}{0.4pt} \\ \rule{2cm}{0.4pt} & \rule{2cm}{0.4pt} & \rule{2cm}{0.4pt} \\ \rule{2cm}{0.4pt} & \rule{2cm}{0.4pt} & \rule{2cm}{0.4pt} \end{bmatrix}$$

6. (10 points) Diagonalize the matrix A: Write $A = PDP^{-1}$ for a diagonal matrix D.

$$\begin{bmatrix} 4 & -1 \\ -3 & 2 \end{bmatrix}$$

$$P = \begin{bmatrix} \_\_\_\_\_\_ & \_\_\_\_\_\_ \\ \_\_\_\_\_\_ & \_\_\_\_\_\_ \end{bmatrix} \quad D = \begin{bmatrix} \_\_\_\_\_\_ & \_\_\_\_\_\_ \\ \_\_\_\_\_\_ & \_\_\_\_\_\_ \end{bmatrix} \quad P^{-1} = \begin{bmatrix} \_\_\_\_\_\_ & \_\_\_\_\_\_ \\ \_\_\_\_\_\_ & \_\_\_\_\_\_ \end{bmatrix}$$

7. (10 points) Use the Gram-Schmidt process to replace the given basis by an orthogonal basis. (It does not have to be orthonormal.)

$$u_1 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \quad u_2 = \begin{bmatrix} 5 \\ 5 \\ 5 \end{bmatrix}, \quad u_3 = \begin{bmatrix} 0 \\ 18 \\ 6 \end{bmatrix}$$

Orthogonal basis: ____

8. (10 points) Determine if the vector $\vec{b}$ is in the column space Col A of the matrix A (circle Yes or No below). If so, find a solution to $A\vec{x} = \vec{b}$. If not, find a *least squares* solution $A\widetilde{x} = \widetilde{b}$.

$$A = \begin{bmatrix} -1 & 2 \\ 1 & 0 \\ 2 & -1 \end{bmatrix}, \qquad \vec{b} = \begin{bmatrix} 0 \\ 4 \\ -1 \end{bmatrix}$$

Is $\vec{b}$ in Col A?

i) Yes　　　ii) No

Answer ($\vec{x}$ or $\widetilde{x}$): $\begin{bmatrix} \underline{\qquad\qquad} \\ \underline{\qquad\qquad} \end{bmatrix}$

9. (8 points) (a) Find the matrix A such that the quadratic form

$$Q(x_1, x_2) = -4x_1^2 + 11x_2^2 - 20x_1x_2$$

comes from the matrix: $Q(\vec{x}) = \vec{x}^T A\vec{x}$.

$$\begin{bmatrix} \_\_\_\_\_\_ & \_\_\_\_\_\_ \\ \_\_\_\_\_\_ & \_\_\_\_\_\_ \end{bmatrix}$$

(b) Change the basis of $\vec{x}$ so that $Q(x_1, x_2)$ is diagonal, i.e., has no cross terms x_1x_2.

New Q: ______________________________________

10. (8 points) Consider two bases

$$\mathcal{C} = \left\{ \begin{bmatrix} -6 \\ -2 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \end{bmatrix} \right\} \qquad \mathcal{B} = \left\{ \begin{bmatrix} -12 \\ -4 \end{bmatrix}, \begin{bmatrix} -15 \\ -6 \end{bmatrix} \right\}$$

for $\mathbb{R}^2$.

Find the change of basis matrix $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ that sends a vector in $\mathcal{B}$ coordinates $[\vec{x}]_{\mathcal{B}}$ to the same vector represented in $\mathcal{C}$ coordinates $[\vec{x}]_{\mathcal{C}}$.

$$\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = \begin{bmatrix} \_\_\_\_\_\_ & \_\_\_\_\_\_ \\ \_\_\_\_\_\_ & \_\_\_\_\_\_ \end{bmatrix}$$